

Regularity of a Weak Solution to the Navier–Stokes Equations via One Component of a Spectral Projection of Vorticity

Jiří Neustupa and Patrick Penel

Abstract

We deal with a weak solution \mathbf{v} to the Navier–Stokes initial value problem in $\mathbb{R}^3 \times (0, T)$. We denote by ω^+ a spectral projection of $\omega \equiv \operatorname{curl} \mathbf{v}$, defined by means of the spectral resolution of identity associated with the self-adjoint operator curl . We show that certain conditions imposed on ω^+ or, alternatively, only on ω_3^+ (the third component of ω^+) imply regularity of solution \mathbf{v} .

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1 Introduction

The Navier–Stokes problem. Let $T > 0$. We denote $Q_T := \mathbb{R}^3 \times (0, T)$. We deal with the Navier–Stokes initial value problem

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \Delta \mathbf{v} \quad \text{in } Q_T, \quad (1.1)$$

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } Q_T, \quad (1.2)$$

$$\mathbf{v}(\mathbf{x}, \cdot) \longrightarrow 0 \quad \text{for } |\mathbf{x}| \rightarrow \infty, \quad (1.3)$$

$$\mathbf{v}(\cdot, 0) = \mathbf{v}_0 \quad \text{in } \mathbb{R}^3 \quad (1.4)$$

for the unknown velocity \mathbf{v} and pressure p . The symbol ν denotes the coefficient of viscosity. It is usually assumed to be a positive constant. Since its value plays no role throughout the paper, we assume that $\nu = 1$.

We assume that \mathbf{v} is a weak solution of the problem (1.1)–(1.4). (This notion was introduced by Leray [13], the exact definition is also given e.g. in [9].) In accordance with [4], we define a *regular point* of solution \mathbf{v} as a point $(\mathbf{x}, t) \in Q_T$ such that there exists a space–time neighbourhood of (\mathbf{x}, t) , where \mathbf{v} is essentially bounded. Points in Q_T that are not regular are called *singular*. The question whether a weak solution can develop a singularity at some time instant $t_0 \in (0, T]$ or if all points $(\mathbf{x}, t) \in Q_T$ are regular points is an important open problem in the theory of the Navier–Stokes equations. There exist many a posteriori criteria, stating that if the weak solution has certain additional properties then it has no singular points (in the whole Q_T or at least in a sub-domain of Q_T). The studies of such criteria have been mainly motivated by Leray [13] (who proved that if the weak solution belongs to the class $L^r(0, T; \mathbf{L}^s(\mathbb{R}^3))$, where $3 < s \leq \infty$ and

$2/r + 3/s = 1$, then it is infinitely differentiable in Q_T) and by Serrin [20] (who proved a certain analog of Leray's criterion, applicable in a sub-domain of Q_T). Exact citations and further details on this topic can be found in the survey paper [9] by Galdi.

On some previous results. Here, we focus on regularity criteria that impose additional conditions on some components of velocity $\mathbf{v} = (v_1, v_2, v_3)$ or its gradient $\nabla \mathbf{v}$ or the corresponding vorticity $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$.

The first result on regularity as a consequence of an a posteriori assumption on one velocity component appeared in [15]: the authors considered the problem in a domain $\Omega \subset \mathbb{R}^3$, assumed that the component v_3 is essentially bounded in a space-time region $D \subset \Omega \times (0, T)$, and proved that \mathbf{v} has no singular points in D . This result has been later successively improved in [16] (v_3 is only supposed to be in $L^r(t_1, t_2; L^s(D'))$ for all sets $D' \times (t_1, t_2) \subset D$ and some $r \in [4, \infty]$, $s \in (6, \infty]$ satisfying $2/r + 3/s \leq \frac{1}{2}$), [17] (the authors generalize the result from [16] and combine an assumption on v_3 with assumptions on v_1, v_2), [12] (v_3 is only assumed to be in $L^r(0, T; L^s(\mathbb{R}^3))$, where $2/r + 3/s = \frac{5}{8}$ for $r \in [\frac{16}{5}, \infty)$ and $s \in (\frac{24}{5}, \infty]$), [6] (the authors consider the spatially periodic problem in \mathbb{R}^3 and use the condition $2/r + 3/s < \frac{2}{3} + 2/(3s)$, $s > \frac{7}{2}$), and [22] (the exponents r, s are supposed to satisfy the conditions $2/r + 3/s \leq \frac{3}{4} + 1/(2s)$, $s > \frac{10}{3}$).

Of a series of papers, where the authors deal with the question of regularity of weak solution \mathbf{v} in dependence on certain integrability properties of some components of the tensor $\nabla \mathbf{v}$, we mention [1], [5], [11], [12], [22], [23] and [19]. In paper [5], the authors prove regularity of solution \mathbf{v} by means of conditions imposed on only two components of vorticity. They assume that the initial velocity \mathbf{v}_0 is “smooth” and $\omega_1, \omega_2 \in L^r(0, T; L^s(\mathbb{R}^3))$ with $1 < r < \infty$, $\frac{3}{2} < s < \infty$, $2/r + 3/s \leq 2$ or the norms of ω_1 and ω_2 in $L^\infty(0, T; L^{3/2}(\mathbb{R}^3))$ are “sufficiently small”. It is a challenging open problem to show whether regularity of weak solution \mathbf{v} can be controlled by only one component of vorticity.

The cited criteria that concern the solution in the whole space hold for any weak solution, while the interior regularity criteria hold for the so called suitable weak solution because here we need to apply an appropriate localization procedure (see e.g. [17], the concept of suitable weak solutions has been introduced in [4]).

The results mentioned above represent attempts to find a minimum quantity which controls regularity of the solution. If such a quantity is in some sense smooth or integrable then the weak solution is smooth. On the other hand, each such quantity necessarily loses smoothness if a singular point shows up. Thus, the criteria contribute to understanding the behaviour of the solution in the neighbourhood of a hypothetical singular point. The presented paper brings results in this field. The quantity, which is assumed to be “smooth” in this paper, is either a certain spectral projection of vorticity or only one component of that spectral projection. The projection is defined by means of the spectral resolution of identity associated with operator \mathbf{curl} , see (1.5). In the case of only one component, we need to impose a stronger condition on its regularity than in the case of all three components, see Theorems 1 and 2.

Notation and auxiliary results. We denote vector functions and spaces of such functions by boldface letters. The norm in $L^q(\mathbb{R}^3)$ (or $\mathbf{L}^q(\mathbb{R}^3)$) is denoted by $\|\cdot\|_{q; \mathbb{R}^3}$. The scalar product in $\mathbf{L}^2(\mathbb{R}^3)$ is denoted by $(\cdot, \cdot)_{2; \mathbb{R}^3}$. The norm in $W^{s,q}(\mathbb{R}^3)$ (or $\mathbf{W}^{s,q}(\mathbb{R}^3)$) is denoted by $\|\cdot\|_{s,q; \mathbb{R}^3}$. Other norms and scalar products are denoted by analogy.

The space $\mathbf{L}_\sigma^2(\mathbb{R}^3)$ is a completion of $\mathbf{C}_{0,\sigma}^\infty(\mathbb{R}^3)$ (the linear space of infinitely differentiable

divergence-free vector functions in \mathbb{R}^3 with a compact support) in $\mathbf{L}^2(\Omega)$. The intersection $\mathbf{W}^{1,2}(\mathbb{R}^3) \cap \mathbf{L}_\sigma^2(\mathbb{R}^3)$ is denoted by $\mathbf{W}_\sigma^{1,2}(\mathbb{R}^3)$. It is a closed subspace of $\mathbf{W}^{1,2}(\mathbb{R}^3)$. Note that $\|\nabla \mathbf{u}\|_{2;\mathbb{R}^3}^2 = \|\mathbf{curl} \mathbf{u}\|_{2;\mathbb{R}^3}^2$ for $\mathbf{u} \in \mathbf{W}_\sigma^{1,2}(\mathbb{R}^3)$. Consequently, $\|\cdot\|_{1,2;\mathbb{R}^3}^2 = \|\cdot\|_{2;\mathbb{R}^3}^2 + \|\mathbf{curl} \cdot\|_{2;\mathbb{R}^3}^2$ in $\mathbf{W}_\sigma^{1,2}(\mathbb{R}^3)$. The dual to $\mathbf{W}_\sigma^{1,2}(\mathbb{R}^3)$ is denoted by $\mathbf{W}_\sigma^{-1,2}(\mathbb{R}^3)$.

The operator $(-\Delta)$, with the domain $W^{2,2}(\mathbb{R}^3)$ (respectively $\mathbf{W}^{2,2}(\mathbb{R}^3)$), is positive and self-adjoint in $L^2(\mathbb{R}^3)$ (respectively in $\mathbf{L}^2(\mathbb{R}^3)$). Its spectrum is purely continuous and covers the non-negative part of the real axis, see e.g. [8].

The Stokes operator $S := \mathbf{curl}^2$, as an operator in $\mathbf{L}_\sigma^2(\mathbb{R}^3)$, coincides with the reduction of $(-\Delta)$ to $\mathbf{L}_\sigma^2(\mathbb{R}^3)$, see e.g. [21, p. 138]. The domain of S is the space $\mathbf{W}^{2,2}(\mathbb{R}^3) \cap \mathbf{L}_\sigma^2(\mathbb{R}^3)$. Operator S is positive, and its spectrum is continuous and covers the interval $[0, \infty)$ on the real axis, see [7] or [8]. The power $S^{1/4}$ of operator S satisfies the Sobolev-type inequality $\|\mathbf{u}\|_{3;\mathbb{R}^3} \leq c_1 \|S^{1/4} \mathbf{u}\|_{2;\mathbb{R}^3}$ for $\mathbf{u} \in D(S^{1/4})$, see [21, p. 141].

Lemma 1. *Operator \mathbf{curl} , with the domain $D(\mathbf{curl}) := \mathbf{W}_\sigma^{1,2}(\mathbb{R}^3)$, is self-adjoint in $\mathbf{L}_\sigma^2(\mathbb{R}^3)$. Its spectrum is continuous and coincides with the whole real axis.*

Proof. Operator \mathbf{curl} maps $\mathbf{W}_\sigma^{1,2}(\mathbb{R}^3)$ into $\mathbf{L}_\sigma^2(\mathbb{R}^3)$. The symmetry of \mathbf{curl} follows from an easy integration by parts. The symmetry means that $\mathbf{curl} \subset \mathbf{curl}^*$, where \mathbf{curl}^* is the adjoint operator to \mathbf{curl} . In order to prove that $\mathbf{curl} = \mathbf{curl}^*$, it is sufficient to show that $D(\mathbf{curl}^*) \subset \mathbf{W}_\sigma^{1,2}(\mathbb{R}^3)$. Thus, let $\mathbf{u} \in D(\mathbf{curl}^*)$. Then there exists $\mathbf{u}^* \in \mathbf{L}_\sigma^2(\mathbb{R}^3)$ such that $(\mathbf{curl} \mathbf{v}, \mathbf{u})_{2;\mathbb{R}^3} = (\mathbf{v}, \mathbf{u}^*)_{2;\mathbb{R}^3}$ for all $\mathbf{v} \in \mathbf{W}_\sigma^{1,2}(\mathbb{R}^3)$. There exists a sequence $\{\mathbf{u}_n\}$ in $\mathbf{W}_\sigma^{1,2}(\mathbb{R}^3)$, converging to \mathbf{u} in the norm of $\mathbf{L}_\sigma^2(\mathbb{R}^3)$. For each $\mathbf{v} \in \mathbf{W}_\sigma^{1,2}(\mathbb{R}^3)$, we have

$$(\mathbf{curl} \mathbf{v}, \mathbf{u})_{2;\mathbb{R}^3} = \lim_{n \rightarrow \infty} (\mathbf{curl} \mathbf{v}, \mathbf{u}_n)_{2;\mathbb{R}^3} = \lim_{n \rightarrow \infty} (\mathbf{v}, \mathbf{curl} \mathbf{u}_n)_{2;\mathbb{R}^3}.$$

Thus, $(\mathbf{v}, \mathbf{u}^*)_{2;\mathbb{R}^3} = \lim_{n \rightarrow \infty} (\mathbf{v}, \mathbf{curl} \mathbf{u}_n)_{2;\mathbb{R}^3}$. This holds, due to the density of $\mathbf{W}_\sigma^{1,2}(\mathbb{R}^3)$ in $\mathbf{L}_\sigma^2(\mathbb{R}^3)$, for all $\mathbf{v} \in \mathbf{L}_\sigma^2(\mathbb{R}^3)$. Hence the sequence $\{\mathbf{curl} \mathbf{u}_n\}$ converges weakly to \mathbf{u}^* in $\mathbf{L}_\sigma^2(\mathbb{R}^3)$. Furthermore, since \mathbf{curl} maps continuously $\mathbf{L}_\sigma^2(\mathbb{R}^3)$ to $\mathbf{W}_\sigma^{-1,2}(\mathbb{R}^3)$, we also have $\mathbf{curl} \mathbf{u}_n \rightarrow \mathbf{curl} \mathbf{u}$ in $\mathbf{W}_\sigma^{-1,2}(\mathbb{R}^3)$. Hence $\mathbf{curl} \mathbf{u} = \mathbf{u}^* \in \mathbf{L}_\sigma^2(\mathbb{R}^3)$. This inclusion, together with the fact that $\mathbf{u} \in \mathbf{L}_\sigma^2(\mathbb{R}^3)$, implies that $\mathbf{u} \in \mathbf{W}_\sigma^{1,2}(\mathbb{R}^3)$. We have proven that operator \mathbf{curl} is self-adjoint in $\mathbf{L}_\sigma^2(\mathbb{R}^3)$.

The spectrum of \mathbf{curl} , which we denote by $\text{Sp}(\mathbf{curl})$, is a subset of the real axis. The residual part is empty, because \mathbf{curl} is self-adjoint. It means that each point $\lambda \in \text{Sp}(\mathbf{curl})$ is either an eigenvalue, or it belongs to $\text{Sp}_c(\mathbf{curl})$ (the continuous spectrum of \mathbf{curl}). If λ is an eigenvalue then λ^2 is an eigenvalue of the Stokes operator S , which is impossible (see e.g. [7, Lemma 2.6]). Thus, $\text{Sp}(\mathbf{curl}) = \text{Sp}_c(\mathbf{curl})$.

Let us finally show that the spectrum covers the whole real axis. All points of $\text{Sp}_c(\mathbf{curl})$ are non-isolated, otherwise they would have been the eigenvalues, see [10, p. 273]. Let $\lambda \in \text{Sp}_c(\mathbf{curl})$, $\lambda \neq 0$. There exists a sequence $\{\mathbf{u}_n\}$ on the unit sphere in $\mathbf{L}_\sigma^2(\mathbb{R}^3)$, such that $\|\mathbf{curl} \mathbf{u}_n - \lambda \mathbf{u}_n\|_{2;\mathbb{R}^3} \rightarrow 0$. Let $\xi \in \mathbb{R}$, $\xi \neq 0$. Put $\mathbf{u}_n^\xi(\mathbf{x}) := \xi^{3/2} \mathbf{u}_n(\xi \mathbf{x})$. Then $\{\mathbf{u}_n^\xi\}$ is a sequence on the unit sphere in $\mathbf{L}_\sigma^2(\mathbb{R}^3)$, satisfying $\|\mathbf{curl} \mathbf{u}_n^\xi - \xi \lambda \mathbf{u}_n^\xi\|_{2;\mathbb{R}^3} \rightarrow 0$. It means that $\xi \lambda$ belongs to $\text{Sp}_c(\mathbf{curl})$ as well. Thus, each real number, different from zero, is in $\text{Sp}_c(\mathbf{curl})$. Since $\text{Sp}_c(\mathbf{curl})$ is closed, we obtain the equality $\text{Sp}_c(\mathbf{curl}) = \mathbb{R}$. \square

Let us note that a self-adjoint realization of operator \mathbf{curl} in an exterior domain, in a more general framework than in the space $\mathbf{L}_\sigma^2(\mathbb{R}^3)$, has been studied in [18].

Let $\{E_\lambda\}$ be the spectral resolution of identity, associated with operator \mathbf{curl} . Projection E_λ is strongly continuous in dependence on λ because $\text{Sp}(\mathbf{curl})$ is continuous, see [10, pp. 353–356]. We denote

$$P^- := E_0 = \int_{-\infty}^0 dE_\lambda \quad \text{and} \quad P^+ := I - E_0 = \int_0^\infty dE_\lambda. \quad (1.5)$$

Operators P^- and P^+ are orthogonal projections in $\mathbf{L}_\sigma^2(\mathbb{R}^3)$ such that $I = P^- + P^+$ and $O = P^- P^+$. We put $\mathbf{L}_\sigma^2(\mathbb{R}^3)^- := P^- \mathbf{L}_\sigma^2(\mathbb{R}^3)$ and $\mathbf{L}_\sigma^2(\mathbb{R}^3)^+ := P^+ \mathbf{L}_\sigma^2(\mathbb{R}^3)$. Both $\mathbf{L}_\sigma^2(\mathbb{R}^3)^-$ and $\mathbf{L}_\sigma^2(\mathbb{R}^3)^+$ are closed subspaces of $\mathbf{L}_\sigma^2(\mathbb{R}^3)$. Operator \mathbf{curl} reduces on each of the spaces $\mathbf{L}_\sigma^2(\mathbb{R}^3)^-$ and $\mathbf{L}_\sigma^2(\mathbb{R}^3)^+$. It is negative on $\mathbf{L}_\sigma^2(\mathbb{R}^3)^-$ and positive on $\mathbf{L}_\sigma^2(\mathbb{R}^3)^+$. We denote by A the operator $|\mathbf{curl}|$, i.e.

$$A := -\mathbf{curl}|_{\mathbf{L}_\sigma^2(\mathbb{R}^3)^-} + \mathbf{curl}|_{\mathbf{L}_\sigma^2(\mathbb{R}^3)^+}. \quad (1.6)$$

Lemma 2. *Operator A is positive, self-adjoint, and $A = S^{1/2}$.*

Proof. Operator A is self-adjoint and positive in each of the spaces $\mathbf{L}_\sigma^2(\mathbb{R}^3)^-$ and $\mathbf{L}_\sigma^2(\mathbb{R}^3)^+$, hence it is self-adjoint and positive in $\mathbf{L}_\sigma^2(\mathbb{R}^3)$ as well. (See also [10, p. 358].)

The formula $A^2 \mathbf{u} = S \mathbf{u}$ clearly holds for $\mathbf{u} \in D(A^2) \cap D(S)$. Clearly, $D(S) \subset D(A^2)$. We claim that the opposite inclusion $D(A^2) \subset D(S)$ is also true: the domain of A^2 is, by definition, the space of all $\mathbf{u} \in \mathbf{W}_\sigma^{1,2}(\mathbb{R}^3)$ such that $A \mathbf{u} \in \mathbf{W}_\sigma^{1,2}(\mathbb{R}^3)$. Using the decomposition $\mathbf{u} = P^- \mathbf{u} + P^+ \mathbf{u}$ and the fact that both the operators A and \mathbf{curl} are reduced on $\mathbf{L}_\sigma^2(\mathbb{R}^3)^-$ and on $\mathbf{L}_\sigma^2(\mathbb{R}^3)^+$, one can verify that $\mathbf{u} \in D(A^2)$ satisfies

$$\|A \mathbf{u}\|_{1,2;\mathbb{R}^3}^2 = \|\mathbf{curl} A \mathbf{u}\|_{2;\mathbb{R}^2}^2 + \|A \mathbf{u}\|_{2;\mathbb{R}^3}^2 = \|\mathbf{curl}^2 \mathbf{u}\|_{2;\mathbb{R}^3}^2 + \|\mathbf{curl} \mathbf{u}\|_{2;\mathbb{R}^3}^2 < \infty.$$

This implies that $\mathbf{u} \in \mathbf{W}^{2,2}(\mathbb{R}^3) \cap \mathbf{W}_\sigma^{1,2}(\mathbb{R}^3) = D(S)$, hence $D(A^2) \subset D(S)$. Consequently, $A^2 = S$.

The resolution of identity associated with operator A is the system of projections $F_\lambda := O$ for $\lambda < 0$, $F_\lambda = E_\lambda - E_{-\lambda}$ for $\lambda > 0$. The family of projections $G_\lambda := O$ for $\lambda < 0$, $G_\lambda := F_{\sqrt{\lambda}}$ for $\lambda > 0$, represents the resolution of identity associated with the operator $A^2 \equiv S$. Operator A can now be expressed in this way:

$$A = \int_0^\infty \lambda dF_\lambda = \int_0^\infty \sqrt{\zeta} dF_{\sqrt{\zeta}} = \int_0^\infty \sqrt{\zeta} dG_\zeta = S^{1/2}.$$

This completes the proof. \square

Another way, how one can prove the identity $A = S^{1/2}$, is the application of Theorem 3.35 in [10]. However, here one needs to verify that both the operators S and A are m -accretive. The identity $A = S^{1/2}$ also follows from [2, Theorem 4, p. 144].

Due to Lemma 2, $A^\alpha = S^{\alpha/2}$ for $\alpha \geq 0$. Consequently,

$$\|\mathbf{u}\|_{3;\mathbb{R}^3} \leq c_1 \|A^{1/2} \mathbf{u}\|_{2;\mathbb{R}^3} \quad (1.7)$$

for $\mathbf{u} \in D(A^{1/2})$.

Recall that $\omega = \mathbf{curl} \mathbf{v}$. We further denote $\mathbf{v}^- := P^- \mathbf{v}$, $\mathbf{v}^+ := P^+ \mathbf{v}$, $\omega^- := P^- \omega$ and $\omega^+ := P^+ \omega$. The components of \mathbf{v}^+ are denoted by v_1^+ , v_2^+ and v_3^+ , the components of functions \mathbf{v}^- , ω^- and ω^+ are denoted by analogy. Since operator \mathbf{curl} commutes with projections P^- and P^+ , we have $\omega^- = \mathbf{curl} \mathbf{v}^- = -A \mathbf{v}^-$ and $\omega^+ = \mathbf{curl} \mathbf{v}^+ = A \mathbf{v}^+$.

As a weak solution of the problem (1.1)–(1.4), \mathbf{v} belongs to $L^2(0, T; \mathbf{W}_\sigma^{1,2}(\mathbb{R}^3)) \cap L^\infty(0, T; \mathbf{L}_\sigma^2(\mathbb{R}^3))$. We say that \mathbf{v} satisfies (EI) (= the energy inequality) if

$$\|\mathbf{v}(t)\|_{2; \mathbb{R}^3}^2 + 2 \int_s^t \|\nabla \mathbf{v}(\tau)\|_{2; \mathbb{R}^3}^2 d\tau \leq \|\mathbf{v}(s)\|_{2; \mathbb{R}^3}^2 \quad (1.8)$$

for $s = 0$ and all $t \in [0, T]$. We say that \mathbf{v} satisfies (SEI) (= the strong energy inequality) if (1.8) holds for a.a. $s \in [0, T]$ and all $t \in [s, T]$.

The next two theorems represent the main results of the paper.

Theorem 1. *Let \mathbf{v} be a weak solution to the problem (1.1)–(1.4). Assume that at least one of the two conditions*

- (i) $(-\Delta)^{1/4} \omega^+ \in \mathbf{L}^2(Q_T)$,
- (ii) $(-\Delta)^{3/4} \omega_3^+ \in L^2(Q_T)$

and at least one of the two conditions

- (a) $\mathbf{v}_0 \in \mathbf{L}_\sigma^2(\mathbb{R}^3)$ and \mathbf{v} satisfies (SEI),
- (b) $\mathbf{v}_0 \in D(A^{1/2})$ and \mathbf{v} satisfies (EI)

hold. Then the norm $\|A^{1/2} \mathbf{v}\|_{2; \mathbb{R}^3}$ is bounded in each time interval (ϑ, T) , where $0 < \vartheta < T$. (If condition (b) holds then $\|A^{1/2} \mathbf{v}\|_{2; \mathbb{R}^3}$ is even bounded on the whole interval $(0, T)$.) Consequently, solution \mathbf{v} has no singular points in Q_T .

The proof of existence of a weak solution to (1.1)–(1.4), satisfying (EI) and (SEI) under the assumption that $\mathbf{v}_0 \in \mathbf{L}_\sigma^2(\mathbb{R}^3)$, is given in [13]. Thus, conditions (a) and (b) do not cause any remarkable restrictions.

The next theorem is a generalization of Theorem 1. Before we formulate it, we introduce some notation. Suppose that $a = a(t)$ is a function in the interval $(0, T)$ with values in $[-\infty, \infty)$. We denote by $a_+(t)$ the positive part and by $a_-(t)$ the negative part of $a(t)$. We put

$$P_{a(t)}^+ := I - E_{a(t)} = \int_{a(t)}^\infty dE_\lambda, \quad (1.9)$$

$\mathbf{v}_a^+(t) := P_{a(t)}^+ \mathbf{v}(t)$, and $\omega_a^+(t) := P_{a(t)}^+ \omega(t) = \mathbf{curl} \mathbf{v}_a^+(t)$. The third component of function ω_a^+ is denoted by ω_{a3}^+ .

Theorem 2. *Let \mathbf{v} be a weak solution to the problem (1.1)–(1.4). Assume that at least one of the two conditions*

- (iii) $a_+ \in L^3(0, T)$ and $(-\Delta)^{1/4} \omega_a^+ \in \mathbf{L}^2(Q_T)$,
- (iv) $a_+ \in L^3(0, T)$, $a_- \in L^5(0, T)$ and $(-\Delta)^{3/4} \omega_{a3}^+ \in L^2(Q_T)$

and at least one of conditions (a) and (b) are fulfilled. Then the statements of Theorem 1 hold.

If $a \equiv 0$ then Theorems 1 and 2 coincide. Theorem 1 is proven in Section 2, the proof of Theorem 2 is the contents of Section 3. Several remarks are postponed to Section 4.

2 Proof of Theorem 1

Throughout this section, we denote by c a generic constant, which is always independent of solution \mathbf{v} . Numbered constants have the same value (also independent of \mathbf{v}) in the whole paper.

Suppose that solution \mathbf{v} satisfies condition (a). Then it also satisfies the assumptions of the so called *Theorème de Structure*, see [9, p. 57]. (The theorem was in fact for the first time formulated by Leray in [13, pp. 244, 245].) Due to this theorem, there exists a system $\{(a_\gamma, b_\gamma)\}_{\gamma \in \Gamma}$ of disjoint open intervals in $(0, T)$ such that the measure of $(0, T) \setminus \cup_{\gamma \in \Gamma} (a_\gamma, b_\gamma)$ is zero, \mathbf{v} is of the class C^∞ on $\mathbb{R}^3 \times (a_\gamma, b_\gamma)$ for all $\gamma \in \Gamma$, and $\|A^{1/2}\mathbf{v}\|_{2; \mathbb{R}^3}$ is locally bounded in each of the intervals (a_γ, b_γ) . If a singularity develops at the time instant b_γ then $\|A^{1/2}\mathbf{v}(t)\|_{2; \mathbb{R}^3} \rightarrow \infty$ for $t \rightarrow b_\gamma^-$. In this case, we call b_γ the *epoch of irregularity*. In order to prove that solution \mathbf{v} has no singular points in Q_T , it is sufficient to show that there are no epochs of irregularity in $(0, T)$. Assume, therefore, that $t \in (a_\gamma, b_\gamma)$ for some fixed $\gamma \in \Gamma$.

The Navier–Stokes equation (1.1) (with $\nu = 1$) can also be written in the equivalent form

$$\partial_t \mathbf{v} + \boldsymbol{\omega} \times \mathbf{v} + \mathbf{curl}^2 \mathbf{v} = -\nabla(p + \frac{1}{2}|\mathbf{v}|^2). \quad (2.1)$$

Multiplying this equation by $A\mathbf{v}$, and integrating in \mathbb{R}^3 , we obtain

$$\frac{d}{dt} \frac{1}{2} \|A^{1/2}\mathbf{v}\|_{2; \mathbb{R}^3}^2 - 2(\boldsymbol{\omega}^+ \times \mathbf{v}, \boldsymbol{\omega}^-)_{2; \mathbb{R}^3} + \|A^{3/2}\mathbf{v}\|_{2; \mathbb{R}^3}^2 = 0. \quad (2.2)$$

We have used the identities

$$\begin{aligned} [\boldsymbol{\omega} \times \mathbf{v}] \cdot A\mathbf{v} &= [(\boldsymbol{\omega}^+ + \boldsymbol{\omega}^-) \times \mathbf{v}] \cdot (\boldsymbol{\omega}^+ - \boldsymbol{\omega}^-) = -[\boldsymbol{\omega}^+ \times \mathbf{v}] \cdot \boldsymbol{\omega}^- + [\boldsymbol{\omega}^- \times \mathbf{v}] \cdot \boldsymbol{\omega}^+ \\ &= -2[\boldsymbol{\omega}^+ \times \mathbf{v}] \cdot \boldsymbol{\omega}^-. \end{aligned}$$

The scalar product $(\boldsymbol{\omega}^+ \times \mathbf{v}, \boldsymbol{\omega}^-)_{2; \mathbb{R}^3}$ can be estimated:

$$\begin{aligned} |(\boldsymbol{\omega}^+ \times \mathbf{v}, \boldsymbol{\omega}^-)_{2; \mathbb{R}^3}| &\leq \|\boldsymbol{\omega}^+\|_{3; \mathbb{R}^3} \|\mathbf{v}\|_{3; \mathbb{R}^3} \|\boldsymbol{\omega}^-\|_{3; \mathbb{R}^3} \\ &\leq c_1^3 \|A^{1/2}\boldsymbol{\omega}^+\|_{2; \mathbb{R}^3} \|A^{1/2}\mathbf{v}\|_{2; \mathbb{R}^3} \|A^{1/2}\boldsymbol{\omega}^-\|_{2; \mathbb{R}^3} \\ &\leq \frac{1}{4} \|A^{1/2}\boldsymbol{\omega}^-\|_{2; \mathbb{R}^3}^2 + c_1^6 \|A^{1/2}\mathbf{v}\|_{2; \mathbb{R}^3}^2 \|A^{1/2}\boldsymbol{\omega}^+\|_{2; \mathbb{R}^3}^2 \\ &\leq \frac{1}{4} \|A^{3/2}\mathbf{v}\|_{2; \mathbb{R}^3}^2 + c_1^6 \|A^{1/2}\mathbf{v}\|_{2; \mathbb{R}^3}^2 \|A^{1/2}\boldsymbol{\omega}^+\|_{2; \mathbb{R}^3}^2. \end{aligned} \quad (2.3)$$

Equation (2.2) and inequalities (2.3) yield

$$\frac{d}{dt} \|A^{1/2}\mathbf{v}\|_{2; \mathbb{R}^3}^2 + \|A^{3/2}\mathbf{v}\|_{2; \mathbb{R}^3}^2 \leq 4c_1^6 \|A^{1/2}\mathbf{v}\|_{2; \mathbb{R}^3}^2 \|A^{1/2}\boldsymbol{\omega}^+\|_{2; \mathbb{R}^3}^2. \quad (2.4)$$

The case of condition (i). If condition (i) of Theorem 1 is fulfilled then the term $\|A^{1/2}\boldsymbol{\omega}^+\|_{2; \mathbb{R}^3}^2$ on the right hand side of (2.4) is in $L^1(0, T)$. Hence we can choose $\tau \in (a_\gamma, b_\gamma)$ and apply Gronwall's inequality to (2.4) on the time interval $[\tau, b_\gamma)$. In this way, we show that $\|A^{1/2}\mathbf{v}\|_{2; \mathbb{R}^3}$ is bounded on the interval $[\tau, b_\gamma)$, which means that b_γ is not an epoch of irregularity.

The case of condition (ii). Let us further assume that condition (ii) of Theorem 1 holds. This case is much more subtle and it is considered in the rest of Section 2. The crucial part of the proof is

the estimate of $\|A^{1/2}\omega^+\|_{2;\mathbb{R}^3}^2$. The next paragraphs head towards this aim. We derive an estimate at a fixed time instant t , hence we mostly omit for brevity writing t among the variables of ω^+ and other related functions. Recall that t is supposed to be in the interval (a_γ, b_γ) , where solution \mathbf{v} is smooth. The function value $\mathbf{v}(t)$ even belongs to $\mathbf{W}^{2,2}(\mathbb{R}^3)$, as follows from [9, Theorem 6.1]. Hence $\omega(t) \in \mathbf{W}_\sigma^{1,2}(\mathbb{R}^3)$ and, consequently, $\omega^+(t)$ also belongs to $\mathbf{W}_\sigma^{1,2}(\mathbb{R}^3)$.

Sets K_ξ^{mn} , C^{mn} and the partition of function ω^+ . In this paragraph, we define sets $K_\xi^{mn} \subset \mathbb{R}^2$, $C^{mn} \subset \mathbb{R}^3$, and we successively introduce auxiliary functions η^{mn} , \mathbf{V}^{mn} , y_{mn}^{kl} and z_{mn}^{kl} (for $m, n \in \mathbb{Z}$ and $k \in \{m-1; m; m+1\}$, $l \in \{n-1; n; n+1\}$).

Let us say in advance that K_2^{mn} is a square in \mathbb{R}^2 with the sides of length 5 and $C^{mn} = K_2^{mn} \times \mathbb{R}$. Using the functions η^{mn} and \mathbf{V}^{mn} , we create a partition of function ω^+ which consists of functions ω^{mn} such that $\text{supp } \omega^{mn} \subset C^{mn}$. In following paragraphs, we derive certain estimates of ω^{mn} (based on estimates of the auxiliary functions η^{mn} , \mathbf{V}^{mn} , y_{mn}^{kl} , z_{mn}^{kl} on sets C^{mn}), which strongly use the structure $C^{mn} = K_2^{mn} \times \mathbb{R}$ of sets C^{mn} and the fact that K_2^{mn} are squares in \mathbb{R}^2 with the length of the sides independent of m, n . Then, using the expansion $\omega^+ = \sum_{m,n \in \mathbb{Z}} \omega^{mn}$, we derive an estimate of $A^{1/2}\omega^+$ which is needed in (2.4). (See estimate (2.17).)

We begin with the definition of sets $K_\xi^{mn} \subset \mathbb{R}^2$ and $C^{mn} \subset \mathbb{R}^3$: for $m, n \in \mathbb{Z}$ and $\xi \in (-\frac{1}{2}, \infty)$, we denote $K_\xi^{mn} := (m - \xi, m + 1 + \xi) \times (n - \xi, n + 1 + \xi)$. Further, we put $C^{mn} := K_2^{mn} \times \mathbb{R} = (m - 2, m + 3) \times (n - 2, n + 3) \times \mathbb{R} \subset \mathbb{R}^3$. Thus, K_ξ^{mn} are squares in \mathbb{R}^2 , while C^{mn} are cylinders in \mathbb{R}^3 .

Let $\epsilon \in (0, \frac{1}{8})$ be fixed. There exists a partition of unity with these properties: the partition consists of the system $\{\eta^{mn}\}_{m,n \in \mathbb{Z}}$ of infinitely differentiable functions of two variables, such that

- a) $\eta^{mn} = 1$ in $K_{-\epsilon}^{mn}$, $\eta^{mn} = 0$ in $\mathbb{R}^2 \setminus K_\epsilon^{mn}$, $0 \leq \eta^{mn} \leq 1$ in \mathbb{R}^2 ,
- b) $\eta^{m+i, n+j}(x_1, x_2) = \eta^{mn}(x_1 + i, x_2 + j)$ for all $i, j \in \mathbb{Z}$,
- c) $\sum_{m,n \in \mathbb{Z}} \eta^{mn} = 1$ in \mathbb{R}^2 .

(Function η^{mn} can be e.g. defined by means of a mollifier with the kernel supported on $B_\epsilon(\mathbf{0})$, applied to the characteristic function of the square K_0^{mn} .)

We denote by ∇_{2D} the 2D nabla operator (∂_1, ∂_2) , and by ω_{2D}^+ the 2D vector field (ω_1^+, ω_2^+) . Applying successively the procedure of solving the equation $\nabla_{2D} \cdot \mathbf{u} = f$, especially the so called Bogovskij formula (see e.g. [3]), we deduce that there exists a system $\{\mathbf{V}^{mn}\}_{m,n \in \mathbb{Z}}$ of 2D vector functions $\mathbf{V}^{mn} = (V_1^{mn}, V_2^{mn})$ defined in \mathbb{R}^3 with the properties

- d) $\nabla_{2D} \cdot \mathbf{V}^{mn} = -\nabla_{2D} \eta^{mn} \cdot \omega_{2D}^+$ in \mathbb{R}^3 ,
- e) $\text{supp } \mathbf{V}^{mn} \subset [K_{2\epsilon}^{mn} \setminus K_{-2\epsilon}^{mn}] \times \mathbb{R}$,
- f) $\sum_{m,n \in \mathbb{Z}} \mathbf{V}^{mn} = \mathbf{0}$ in \mathbb{R}^3 ,
- g) $\|\mathbf{V}^{mn}\|_{2;C^{mn}} + \|\nabla_{2D} \mathbf{V}^{mn}\|_{2;C^{mn}} \leq c \|\omega_{2D}^+\|_{2;C^{mn}}$,
- h) $\|\partial_3 \mathbf{V}^{mn}\|_{2;C^{mn}} \leq c \|\partial_3 \omega_{2D}^+\|_{2;C^{mn}}$.

Constant c is always independent of m and n . We can derive from the last two estimates, by interpolation, that

$$\|\mathbf{V}^{mn}\|_{1/2,2;C^{mn}} \leq c \|\omega^+\|_{1/2,2;C^{mn}}. \quad (2.5)$$

For technical reasons, we put $V_3^{mn} := 0$ and we further consider \mathbf{V}^{mn} to be the 3D vector field. Further, we put

$$\omega^{mn} := \eta^{mn} \omega^+ - \mathbf{V}^{mn}.$$

The components of ω^{mn} are denoted by ω_1^{mn} , ω_2^{mn} and ω_3^{mn} . By analogy with ω_{2D}^+ , we also denote $\omega_{2D}^{mn} := (\omega_1^{mn}, \omega_2^{mn})$. Function ω^{mn} is divergence-free in \mathbb{R}^3 , it equals ω^+ in $K_{-2\epsilon}^{mn} \times \mathbb{R}$, and its support is a subset of $K_{2\epsilon}^{mn} \times \mathbb{R}$. Moreover, we have $\omega^+ = \sum_{m,n \in \mathbb{Z}} \omega^{mn}$.

The term $\|A^{1/2} \omega^+\|_{2; \mathbb{R}^3}^2$ can now be written in this form:

$$\begin{aligned} \|A^{1/2} \omega^+\|_{2; \mathbb{R}^3}^2 &= (A \omega^+, \omega^+)_{2; \mathbb{R}^3} = (\mathbf{curl} \omega^+, \omega^+)_{2; \mathbb{R}^3} = \sum_{m,n \in \mathbb{Z}} \sum_{k,l \in \mathbb{Z}} (\mathbf{curl} \omega^{mn}, \omega^{kl})_{2; \mathbb{R}^3} \\ &= \sum_{m,n \in \mathbb{Z}} \sum_{\substack{k \in \{m-1; m; m+1\} \\ l \in \{n-1; n; n+1\}}} (\mathbf{curl} \omega^{mn}, \omega^{kl})_{2; C^{mn}}. \end{aligned} \quad (2.6)$$

The last equality holds because the supports of ω^{mn} and ω^{kl} have a non-empty intersection only if $k \in \{m-1; m; m+1\}$ and $l \in \{n-1; n; n+1\}$. In this case, both the supports are subsets of C^{mn} .

Operator $(-\Delta)_{mn}$. We denote by $(-\Delta)_{mn}$ the operator $-\Delta$ with the domain $D((-\Delta)_{mn}) := W^{2,2}(C^{mn}) \cap W_0^{1,2}(C^{mn})$. Operator $(-\Delta)_{mn}$ is positive and self-adjoint in $L^2(C^{mn})$, with a bounded inverse. The powers of $(-\Delta)_{mn}$, with positive as well as negative exponents, can be defined in the usual way by means of the corresponding spectral expansion, see e.g. [10].

Auxiliary functions y_{mn}^{kl} . We denote by y_{mn}^{kl} the solution of the 2D Neumann problem

$$\Delta_{2D} y_{mn}^{kl} = -(-\Delta)_{mn}^{1/4} (\partial_3 \omega_3^{kl}) \text{ in } K_2^{mn}, \quad \frac{\partial y_{mn}^{kl}}{\partial \mathbf{n}} = 0 \text{ on } \partial K_2^{mn} \quad (2.7)$$

for $m, n \in \mathbb{Z}$, $k \in \{m-1; m; m+1\}$ and $l \in \{n-1; n; n+1\}$. Function y_{mn}^{kl} satisfies the estimate

$$\|\nabla_{2D} y_{mn}^{kl}\|_{2; K_2^{mn}}^2 + \|\nabla_{2D}^2 y_{mn}^{kl}\|_{2; K_2^{mn}}^2 \leq c \|(-\Delta)_{mn}^{1/4} (\partial_3 \omega_3^{kl})\|_{2; K_2^{mn}}^2, \quad (2.8)$$

where c is independent of m, n, k and l . Since $\partial_3 \omega_3^{kl}$ is a function of three variables x_1, x_2, x_3 , function y_{mn}^{kl} naturally depends not only on x_1, x_2 , but also on x_3 . Integrating the last estimate with respect to x_3 , we obtain

$$\|\nabla_{2D}^2 y_{mn}^{kl}\|_{2; C^{mn}}^2 + \|\nabla_{2D} y_{mn}^{kl}\|_{2; C^{mn}}^2 \leq c \|(-\Delta)_{mn}^{1/4} \partial_3 \omega_3^{kl}\|_{2; C^{mn}}^2. \quad (2.9)$$

Auxiliary functions z_{mn}^{kl} . We define function z_{mn}^{kl} to be the solution of the equation

$$\nabla_{2D}^\perp z_{mn}^{kl} = (-\Delta)_{mn}^{1/4} \omega_{2D}^{kl} - \nabla_{2D} y_{mn}^{kl} \quad (2.10)$$

in K_2^{mn} . (Here, we denote by ∇_{2D}^\perp the operator $(-\partial_2, \partial_1)$.) The solution exists because

$$\nabla_{2D} \cdot [(-\Delta)_{mn}^{1/4} \omega_{2D}^{kl} - \nabla_{2D} y_{mn}^{kl}] = 0.$$

Solution z_{mn}^{kl} depends not only on x_1, x_2 , but also on x_3 because the right hand side of equation (2.10) depends on x_3 as well. Function z_{mn}^{kl} is the so called *stream function* of the 2D vector field $(-\Delta)_{mn}^{1/4} \omega_{2D}^{kl} - \nabla_{2D} y_{mn}^{kl}$. For each fixed $x_3 \in \mathbb{R}$, z_{mn}^{kl} satisfies the estimate

$$\|\nabla_{2D} z_{mn}^{kl}\|_{2; K_2^{mn}} \leq c (\|(-\Delta)_{mn}^{1/4} \omega_{2D}^{kl}\|_{2; K_2^{mn}} + \|\nabla_{2D} y_{mn}^{kl}\|_{2; K_2^{mn}}). \quad (2.11)$$

Moreover, z_{mn}^{kl} is constant on ∂C^{mn} ($= \partial K_2^{mn} \times \mathbb{R}$). This follows from the identities

$$\nabla_{2D}^\perp z_{mn}^{kl} \cdot \mathbf{n} = (-\Delta)_{mn}^{1/4} \omega_{2D}^{kl} \cdot \mathbf{n} - \nabla_{2D} y_{mn}^{kl} \cdot \mathbf{n} = 0,$$

valid on ∂C^{mn} . Indeed, the second term $\nabla_{2D} y_{mn}^{kl} \cdot \mathbf{n}$ equals zero on ∂C^{mn} by definition of y_{mn}^{kl} . The first term $(-\Delta)_{mn}^{1/4} \omega_{2D}^{kl}$ is zero on ∂C^{mn} because $\omega^{mn} \in D((-\Delta)_{mn})$, hence $(-\Delta)_{mn}^{1/4} \omega^{mn} \in D((-\Delta)_{mn}^{3/4})$, and functions from $D((-\Delta)_{mn}^{3/4})$ have the trace on ∂C^{mn} equal to zero. (This can be easily verified because $D((-\Delta)_{mn}^{1/2}) = W_0^{1,2}(C^{mn})$, which implies that $D((-\Delta)_{mn}^{3/4})$ is the interpolation space between $D((-\Delta)_{mn}) \equiv W^{2,2}(C^{mn}) \cap W_0^{1,2}(C^{mn})$ and $W_0^{1,2}(C^{mn})$, and both the spaces contain only functions whose traces are equal to zero on ∂C^{mn} .) Function z_{mn}^{kl} is unique up to an additive function of t and x_3 . We can now choose this function so that $z_{mn}^{kl} = 0$ on ∂C^{mn} . This choice, together with (2.11) and (2.9), implies that

$$\begin{aligned} \|z_{mn}^{kl}\|_{2; C^{mn}} &\leq c (\|(-\Delta)_{mn}^{1/4} \omega_{2D}^{kl}\|_{2; C^{mn}} + \|\nabla_{2D} y_{mn}^{kl}\|_{2; C^{mn}}) \\ &\leq c (\|(-\Delta)_{mn}^{1/4} \omega^{kl}\|_{2; C^{mn}} + \|(-\Delta)_{mn}^{1/4} (\partial_3 \omega_3^{kl})\|_{2; C^{mn}}). \end{aligned} \quad (2.12)$$

The estimate of $(\mathbf{curl} \omega^{mn}, \omega^{kl})_{2; C^{mn}}$. We denote $\mathbf{w}^{mn} \equiv (w_1^{mn}, w_2^{mn}, w_3^{mn}) := \mathbf{curl} \omega^{mn}$ and $\mathbf{w}_{2D}^{mn} := (w_1^{mn}, w_2^{mn})$. We always assume that $k \in \{m-1; m; m+1\}$ and $l \in \{n-1; n; n+1\}$. Due to the definition of functions y_{mn}^{kl} and z_{mn}^{kl} , function $(-\Delta)_{mn}^{1/4} \omega^{kl}$ has the form

$$(-\Delta)_{mn}^{1/4} \omega^{kl} = \begin{pmatrix} \partial_1 y_{mn}^{kl} \\ \partial_2 y_{mn}^{kl} \\ (-\Delta)_{mn}^{1/4} \omega_3^{kl} \end{pmatrix} + \mathbf{curl} \begin{pmatrix} 0 \\ 0 \\ z_{mn}^{kl} \end{pmatrix} \quad \text{in } C^{mn}.$$

Hence

$$\begin{aligned} (\mathbf{curl} \omega^{mn}, \omega^{kl})_{2; C^{mn}} &= (\mathbf{w}^{mn}, \omega^{kl})_{2; C^{mn}} = \int_{C^{mn}} (-\Delta)_{mn}^{-1/4} \mathbf{w}^{mn} \cdot (-\Delta)_{mn}^{1/4} \omega^{kl} \, d\mathbf{x} \\ &= \int_{C^{mn}} \left[(-\Delta)_{mn}^{-1/4} \mathbf{w}^{mn} \cdot \begin{pmatrix} \partial_1 y_{mn}^{kl} \\ \partial_2 y_{mn}^{kl} \\ (-\Delta)_{mn}^{1/4} \omega_3^{kl} \end{pmatrix} + (-\Delta)_{mn}^{-1/4} \mathbf{curl}^2 \omega^{mn} \cdot \begin{pmatrix} 0 \\ 0 \\ z_{mn}^{kl} \end{pmatrix} \right] d\mathbf{x} \\ &= \int_{C^{mn}} \left[(-\Delta)_{mn}^{-1/4} \mathbf{w}^{mn} \cdot \begin{pmatrix} \partial_1 y_{mn}^{kl} \\ \partial_2 y_{mn}^{kl} \\ (-\Delta)_{mn}^{1/4} \omega_3^{kl} \end{pmatrix} + (-\Delta)_{mn}^{3/4} \omega_3^{mn} z_{mn}^{kl} \right] d\mathbf{x} \\ &= \int_{C^{mn}} \{ (-\Delta)_{mn}^{-1/4} \mathbf{w}_{2D}^{mn} \cdot \nabla_{2D} y_{mn}^{kl} + (-\Delta)_{mn}^{-1/4} w_3^{mn} (-\Delta)_{mn}^{1/4} \omega_3^{kl} + (-\Delta)_{mn}^{3/4} \omega_3^{mn} z_{mn}^{kl} \} d\mathbf{x} \\ &\leq c \|\omega^{mn}\|_{1/2, 2; C^{mn}} \|\nabla_{2D} y_{mn}^{kl}\|_{2; C^{mn}} + c \|\omega^{mn}\|_{1/2, 2; C^{mn}} \|\omega_3^{kl}\|_{1/2, 2; C^{mn}} \\ &\quad + \|\omega_3^{mn}\|_{3/4, 2; C^{mn}} \|z_{mn}^{kl}\|_{2; C^{mn}} \\ &\leq c \|\omega^{mn}\|_{1/2, 2; C^{mn}} \|(-\Delta)_{mn}^{1/4} (\partial_3 \omega_3^{kl})\|_{2; C^{mn}} + c \|\omega^{mn}\|_{1/2, 2; C^{mn}} \|\omega_3^{kl}\|_{1/2, 2; C^{mn}} \\ &\quad + c \|\omega_3^{mn}\|_{3/2, 2; C^{mn}} (\|(-\Delta)_{mn}^{1/4} \omega^{kl}\|_{2; C^{mn}} + \|(-\Delta)_{mn}^{1/4} (\partial_3 \omega_3^{kl})\|_{2; C^{mn}}). \end{aligned} \quad (2.13)$$

Each term on the right hand side contains some norm of $\omega_3^{mn} (= \eta^{mn} \omega_3^+)$ or $\omega_3^{kl} (= \eta^{kl} \omega_3^+)$. This is how the third component ω_3^+ controls the scalar product $(\text{curl } \omega^{mn}, \omega^{kl})_{2; C^{mn}}$. The right hand side of (2.13) is further less than or equal to

$$\begin{aligned}
& c \|\omega^{mn}\|_{1/2,2; C^{mn}} \|\omega_3^{kl}\|_{3/2,2; C^{mn}} + c \|\omega_3^{mn}\|_{3/2,2; C^{mn}} \|\omega^{kl}\|_{3/2,2; C^{mn}} \\
& + c \|\omega_3^{mn}\|_{3/2,2; C^{mn}} \|\omega_3^{kl}\|_{3/2,2; C^{mn}} \\
& \leq \delta \|\omega^{mn}\|_{1/2,2; C^{mn}}^2 + \delta \|\omega^{kl}\|_{1/2,2; C^{mn}}^2 + c(\delta) \|\omega_3^{mn}\|_{3/2,2; C^{mn}}^2 + c(\delta) \|\omega_3^{kl}\|_{3/2,2; C^{mn}}^2 \\
& \leq \delta \|\eta^{mn} \omega^+\|_{1/2,2; C^{mn}}^2 + \delta \|\mathbf{V}^{mn}\|_{1/2,2; C^{mn}}^2 + \delta \|\eta^{kl} \omega^+\|_{1/2,2; C^{mn}}^2 + \delta \|\mathbf{V}^{kl}\|_{1/2,2; C^{mn}}^2 \\
& + c(\delta) \|\eta^{mn} \omega_3^+\|_{3/2,2; C^{mn}}^2 + c(\delta) \|\eta^{kl} \omega_3^+\|_{3/2,2; C^{mn}}^2. \tag{2.14}
\end{aligned}$$

The norm $\|\mathbf{V}^{mn}\|_{1/2,2; C^{mn}}$ can be estimated by means of (2.5). Since \mathbf{V}^{kl} is supported inside C^{mn} , one can also derive (by analogy with (2.5)) that $\|\mathbf{V}^{kl}\|_{1/2,2; C^{mn}} \leq c \|\omega^+\|_{1/2,2; C^{mn}}$. Furthermore the norm $\|\eta^{mn} \omega^+\|_{1/2,2; C^{mn}}$ can be estimated by $c \|\omega^+\|_{1/2,2; C^{mn}}$. (This can be easily proven in the same way as Theorem I.7.3 in [14].) The other terms on the right hand side of (2.14) that contain functions η^{mn} or η^{kl} can be estimated similarly. Thus, (2.14) yields

$$(\text{curl } \omega^{mn}, \omega^{kl})_{2; C^{mn}} \leq \delta c \|\omega^+\|_{1/2,2; C^{mn}}^2 + c(\delta) \|\omega_3^+\|_{3/2,2; C^{mn}}^2. \tag{2.15}$$

The estimate of the right hand side of (2.6). The sum $\sum_{m,n \in \mathbb{Z}}$ in (2.6) can be split to twenty five parts, which successively contain the sums over $m = 0 \bmod 5, \dots, m = 4 \bmod 5$ and $n = 0 \bmod 5, \dots, n = 4 \bmod 5$.

Let us consider e.g. the case $m, n \in \mathbb{Z}, m = 0 \bmod 5, n = 0 \bmod 5$ (i.e. m and n are integer multiples of 5). Denote the sum over these m, n by $\sum_{m,n \in \mathbb{Z}}^{(1)}$, and the sums over twenty four other possibilities by $\sum_{m,n \in \mathbb{Z}}^{(2)}, \dots, \sum_{m,n \in \mathbb{Z}}^{(25)}$. The cylinders C^{mn} corresponding to the first case are disjoint and their union equals \mathbb{R}^3 up to the set of measure zero. Applying (2.15), we have

$$\begin{aligned}
& \sum_{m,n \in \mathbb{Z}}^{(1)} \sum_{\substack{k \in \{m-1; m; m+1\} \\ l \in \{n-1; n; n+1\}}} (\text{curl } \omega^{mn}, \omega^{kl})_{2; C^{mn}} \\
& \leq \delta c \sum_{m,n \in \mathbb{Z}}^{(1)} \|\omega^+\|_{1/2,2; C^{mn}}^2 + c(\delta) \sum_{m,n \in \mathbb{Z}}^{(1)} \|\omega_3^+\|_{3/2,2; C^{mn}}^2. \tag{2.16}
\end{aligned}$$

Obviously, the L^2 -norms and $W^{1,2}$ -norms of ω^+ satisfy the identities

$$\sum_{m,n \in \mathbb{Z}}^{(1)} \|\omega^+\|_{2; C^{mn}}^2 = \|\omega^+\|_{2; \mathbb{R}^3}^2 \quad \text{and} \quad \sum_{m,n \in \mathbb{Z}}^{(1)} \|\omega^+\|_{1,2; C^{mn}}^2 = \|\omega^+\|_{1,2; \mathbb{R}^3}^2.$$

Applying appropriately the theorem on interpolation (see [14, Theorem I.5.1]), we derive that

$$\sum_{m,n \in \mathbb{Z}}^{(1)} \|\omega^+\|_{1/2,2; C^{mn}}^2 \leq c \|\omega^+\|_{1/2,2; \mathbb{R}^3}^2.$$

The norms $\|\omega_3^+\|_{3/2,2; C^{mn}}$ and $\|\omega_3^+\|_{3/2,2; \mathbb{R}^3}$ satisfy the same inequalities. Applying these inequalities, and estimating the sums $\sum_{m,n \in \mathbb{Z}}^{(2)}, \dots, \sum_{m,n \in \mathbb{Z}}^{(25)}$ in the same way as the sum in (2.16), we get

$$\begin{aligned}
\|A^{1/2}\omega^+\|_{2;\mathbb{R}^3}^2 &\leq \sum_{m,n \in \mathbb{Z}} \sum_{\substack{k \in \{m-1; m; m+1\} \\ l \in \{n-1; n; n+1\}}} (\mathbf{curl} \omega^{mn}, \omega^{kl})_{2; C^{mn}} \\
&\leq \delta c \|\omega^+\|_{1/2,2;\mathbb{R}^3}^2 + c(\delta) \|\omega_3^+\|_{3/2,2;\mathbb{R}^3}^2.
\end{aligned}$$

The first term on the right hand side is less than or equal to $\delta c (\|\omega^+\|_{2;\mathbb{R}^3}^2 + \|A^{1/2}\omega^+\|_{2;\mathbb{R}^3}^2)$. Choosing $\delta > 0$ so small that $\delta c \leq \frac{1}{2}$, and estimating $\|\omega_3^+\|_{3/2,2;\mathbb{R}^3}^2$ from above by $\|\omega_3^+\|_{2;\mathbb{R}^3}^2 + \|(-\Delta)^{3/4}\omega_3^+\|_{2;\mathbb{R}^3}^2$, we finally obtain

$$\|A^{1/2}\omega^+\|_{2;\mathbb{R}^3}^2 \leq c_3 \|\omega^+\|_{2;\mathbb{R}^3}^2 + c_4 \|(-\Delta)^{3/4}\omega_3^+\|_{2;\mathbb{R}^3}^2. \quad (2.17)$$

Completion of the proof. Substituting estimate (2.17) to (2.4), we get

$$\begin{aligned}
&\frac{d}{dt} \frac{1}{2} \|A^{1/2}\mathbf{v}\|_{2;\mathbb{R}^3}^2 + \|A^{3/2}\mathbf{v}\|_{2;\mathbb{R}^3}^2 \\
&\leq 4c_1^6 \|A^{1/2}\mathbf{v}\|_{2;\mathbb{R}^3}^2 (c_3 \|\omega^+\|_{2;\mathbb{R}^3}^2 + c_4 \|(-\Delta)^{3/4}\omega_3^+\|_{2;\mathbb{R}^3}^2).
\end{aligned} \quad (2.18)$$

Recall that this inequality holds for $t \in (a_\gamma, b_\gamma)$. The expression in parentheses on the right hand side is integrable as a function of t in $(0, T)$. Thus, we can again choose $\tau \in (a_\gamma, b_\gamma)$ and apply Gronwall's inequality to (2.18) on the interval $[\tau, b_\gamma)$. This is how we show that $\|A^{1/2}\mathbf{v}\|_{2;\mathbb{R}^3}$ is bounded on $[\tau, b_\gamma)$. Consequently, b_γ cannot be the epoch of irregularity of solution \mathbf{v} and $\|A^{1/2}\mathbf{v}\|_{2;\mathbb{R}^3}$ is therefore bounded on $[\tau, T)$. Since τ can be chosen arbitrarily close to 0, we have proven that $\|A^{1/2}\mathbf{v}\|_{2;\mathbb{R}^3}$ is bounded on each interval of the type (ϑ, T) for $0 < \vartheta < T$.

If solution \mathbf{v} satisfies condition (b) of Theorem 1 then the initial velocity \mathbf{v}_0 belongs to the space $D(A^{1/2})$. Hence there exists $T^* \in (0, T]$ and a strong solution \mathbf{v}^* of the problem (1.1)–(1.4), whose norm $\|A^{1/2}\mathbf{v}^*\|_{2;\mathbb{R}^3}$ is locally bounded on $[0, T^*)$. (See e.g. [21, Section V.4].) The considered weak solution \mathbf{v} coincides with \mathbf{v}^* on $(0, T^*)$ by the theorem on uniqueness, see [9, Theorem 4.2]. (This is the point where we use the fact that \mathbf{v} satisfies (EI).) The time instant T^* is either an epoch of irregularity (if $\|A^{1/2}\mathbf{v}(t)\|_{2;\mathbb{R}^3} \rightarrow \infty$ for $t \rightarrow T^*-$) or $T^* = T$ and $\|A^{1/2}\mathbf{v}\|_{2;\mathbb{R}^3}$ is bounded on $(0, T)$. Repeating the procedure from the previous paragraphs, we can show that T^* cannot be the epoch of irregularity. Thus, $\|A^{1/2}\mathbf{v}\|_{2;\mathbb{R}^3}$ is bounded on $(0, T)$ and solution \mathbf{v} has therefore no singular points in Q_T .

3 Proof of Theorem 2

We can at first copy the proof of Theorem 1 in Section 2 up to inequality (2.4). Instead of “the case of condition (i)”, we consider “the case of condition (iii)”. Recall that F_λ (the resolution of identity associated with operator A) is, for $\lambda \geq 0$, related to E_λ (the resolution of identity associated with operator \mathbf{curl}) by the formula $F_\lambda = E_\lambda - E_{-\lambda}$. Thus, for $t \in (a_\gamma, b_\gamma)$ we have

$$\begin{aligned}
\|A^{1/2}\omega^+(t)\|_{2;\mathbb{R}^3}^2 &= (A\omega^+(t), \omega^+(t))_{2;\mathbb{R}^3} = \int_0^\infty \lambda \, d(F_\lambda \omega^+(t), \omega^+(t))_{2;\mathbb{R}^3} \\
&= \int_0^\infty \lambda \, d((E_\lambda - E_{-\lambda})\omega^+(t), \omega^+(t))_{2;\mathbb{R}^3} = \int_0^\infty \lambda \, d(E_\lambda \omega^+(t), \omega^+(t))_{2;\mathbb{R}^3} \\
&= \int_0^{a_+(t)} \lambda \, d(E_\lambda A\mathbf{v}^+(t), A\mathbf{v}^+(t))_{2;\mathbb{R}^3} + \int_{a_+(t)}^\infty \lambda \, d(E_\lambda \omega^+(t), \omega^+(t))_{2;\mathbb{R}^3}. \quad (3.1)
\end{aligned}$$

(We have used the identity $E_{-\lambda}\omega^+(t) = \mathbf{0}$ for $\lambda \geq 0$.) As in Section 2, we further omit writing (t) . The first integral on the right hand side of (3.1) equals

$$\int_0^{a_+} \lambda^3 \, d(E_\lambda \mathbf{v}^+, \mathbf{v}^+)_{2; \mathbb{R}^3} \leq a_+^3 \int_0^{a_+} d(E_\lambda \mathbf{v}^+, \mathbf{v}^+)_{2; \mathbb{R}^3} \leq a_+^3 \|\mathbf{v}^+\|_{2; \mathbb{R}^3}^2 \leq c_5 a_+^3, \quad (3.2)$$

where c_5 is the essential upper bound of $\|\mathbf{v}\|_{2; \mathbb{R}^3}^2$ on $(0, T)$.

Let us now deal with the second integral on the right hand side of (3.1). If $a \geq 0$ then ω^+ can be expressed as the sum $\omega_{(0,a)} + \omega_a^+$, where $\omega_{(0,a)} := \mathbf{curl} \, \mathbf{v}_{(0,a)}$ and $\mathbf{v}_{(0,a)} := \int_0^a dE_\lambda \mathbf{v} = (E_a - E_0)\mathbf{v}$. Thus, $E_\lambda \omega^+$ (for $\lambda \geq a = a_+$) equals $E_\lambda \omega_{(0,a)} + E_\lambda \omega_a^+ = \omega_{(0,a)} + E_\lambda \omega_a^+$. The differential of $(E_\lambda \omega^+, \omega^+)_{2; \mathbb{R}^3}$ with respect to variable λ is

$$\begin{aligned} d(E_\lambda \omega^+, \omega^+)_{2; \mathbb{R}^3} &= d(\omega_{(0,a)}, \omega^+)_{2; \mathbb{R}^3} + d(E_\lambda \omega_a^+, \omega^+)_{2; \mathbb{R}^3} = d(E_\lambda \omega_a^+, \omega^+)_{2; \mathbb{R}^3} \\ &= d(E_\lambda \omega_a^+, \omega_a^+)_{2; \mathbb{R}^3} + d(E_\lambda \omega_a^+, \omega_{(0,a)})_{2; \mathbb{R}^3} = d(E_\lambda \omega_a^+, \omega_a^+)_{2; \mathbb{R}^3}. \end{aligned}$$

(The last equality holds because $E_\lambda \omega_a^+$ and $\omega_{(0,a)}$ are orthogonal in $L^2(\mathbb{R}^3)$.) Hence

$$\begin{aligned} \int_{a_+}^\infty \lambda \, d(E_\lambda \omega^+, \omega^+)_{2; \mathbb{R}^3} &= \int_{a_+}^\infty \lambda \, d(E_\lambda \omega_a^+, \omega_a^+)_{2; \mathbb{R}^3} = \int_{a_+}^\infty \lambda \, d(F_\lambda \omega_a^+, \omega_a^+)_{2; \mathbb{R}^3} \\ &= \|A^{1/2} \omega_a^+\|_{2; \mathbb{R}^3}^2. \end{aligned} \quad (3.3)$$

Similarly, if $a < 0$ then $\omega_a^+ = \omega_{(a,0)} + \omega^+$, where $\omega_{(a,0)} := \mathbf{curl} \, \mathbf{v}_{(a,0)}$ and $\mathbf{v}_{(a,0)} := \int_a^0 dE_\lambda \mathbf{v} = (E_0 - E_a)\mathbf{v}$. For $\lambda \geq 0$, we have $E_\lambda \omega_a^+ = E_\lambda \omega_{(a,0)} + E_\lambda \omega^+ = \omega_{(a,0)} + E_\lambda \omega^+$ and

$$\begin{aligned} d(E_\lambda \omega_a^+, \omega_a^+)_{2; \mathbb{R}^3} &= d(\omega_{(a,0)}, \omega_a^+)_{2; \mathbb{R}^3} + d(E_\lambda \omega^+, \omega_a^+)_{2; \mathbb{R}^3} = d(E_\lambda \omega^+, \omega_a^+)_{2; \mathbb{R}^3} \\ &= d(E_\lambda \omega^+, \omega_{(a,0)})_{2; \mathbb{R}^3} + d(E_\lambda \omega^+, \omega^+)_{2; \mathbb{R}^3} = d(E_\lambda \omega^+, \omega^+)_{2; \mathbb{R}^3}. \end{aligned}$$

Hence

$$\begin{aligned} \int_{a_+}^\infty \lambda \, d(E_\lambda \omega^+, \omega^+)_{2; \mathbb{R}^3} &= \int_0^\infty \lambda \, d(E_\lambda \omega_a^+, \omega_a^+)_{2; \mathbb{R}^3} \\ &= \int_0^\infty \lambda \, d(F_\lambda \omega_a^+, \omega_a^+)_{2; \mathbb{R}^3} + \int_0^{-a} \lambda \, d(E_{-\lambda} \omega_a^+, \omega_a^+)_{2; \mathbb{R}^3} \\ &= \|A^{1/2} \omega_a^+\|_{2; \mathbb{R}^3}^2 + \int_0^{-a} \lambda \, d(E_{-\lambda} \omega_a^+, \omega_a^+)_{2; \mathbb{R}^3} \\ &= \|A^{1/2} \omega_a^+\|_{2; \mathbb{R}^3}^2 + \int_0^{-a} (-\zeta) \, d(E_\zeta \omega_a^+, \omega_a^+)_{2; \mathbb{R}^3} \leq \|A^{1/2} \omega_a^+\|_{2; \mathbb{R}^3}^2. \end{aligned} \quad (3.4)$$

We observe from (3.3) and (3.4) that for any value of a , the second integral on the right hand of (3.1) is less than or equal to $\|A^{1/2} \omega_a^+\|_{2; \mathbb{R}^3}^2$. Thus, applying also (3.2), we obtain

$$\|A^{1/2} \omega^+\|_{2; \mathbb{R}^3}^2 \leq c_5 a_+^3 + \|A^{1/2} \omega_a^+\|_{2; \mathbb{R}^3}^2. \quad (3.5)$$

Condition (iii) of Theorem 2 implies that the right hand side of (3.5) is integrable on the interval $(0, T)$. The proof of Theorem 2 can now be completed in the same way as the proof of Theorem 1 in the paragraph “the case of condition (i)” in Section 2.

Let us further assume that condition (iv) holds. Let us at first suppose that $a \geq 0$, i.e. $a = a_+$. In order to estimate $\|A^{1/2}\omega_a^+\|_{2;\mathbb{R}^3}$, we can copy the proof of Theorem 1 from “the case of condition (ii)” (which is now replaced by “the case of condition (iv)”) up to (2.17); we only consider ω_a^+ instead of ω^+ and ω_{a3}^+ instead of ω_3^+ . By analogy with (2.17), we obtain

$$\|A^{1/2}\omega_a^+\|_{2;\mathbb{R}^3}^2 \leq c_3 \|\omega_a^+\|_{2;\mathbb{R}^3}^2 + c_4 \|(-\Delta)^{3/4}\omega_{a3}^+\|_{2;\mathbb{R}^3}^2. \quad (3.6)$$

Inequalities (3.5) and (3.6) yield

$$\|A^{1/2}\omega^+\|_{2;\mathbb{R}^3}^2 \leq c_5 a_+^3 + c_3 \|\omega_a^+\|_{2;\mathbb{R}^3}^2 + c_4 \|(-\Delta)^{3/4}\omega_{a3}^+\|_{2;\mathbb{R}^3}^2. \quad (3.7)$$

Further, we suppose that $a < 0$. Now, estimate (3.6) is not true due to this reason: the derivation of (3.6) requires the identity

$$(A\omega_a^+, \omega_a^+)_{2;\mathbb{R}^3} = (\text{curl } \omega_a^+, \omega_a^+)_{2;\mathbb{R}^3}, \quad (3.8)$$

analogous to the identity $(A\omega^+, \omega^+)_{2;\mathbb{R}^3} = (\text{curl } \omega^+, \omega^+)_{2;\mathbb{R}^3}$, which was used in (2.6) and which lead to (2.17). However, while (3.8) holds in the case $a \geq 0$, it does not hold for $a < 0$ (which we now assume). Thus, we begin the estimation of $\|A^{1/2}\omega^+\|_{2;\mathbb{R}^3}^2$ from (2.17). In order to estimate the term $\|(-\Delta)^{3/4}\omega_3^+\|_{2;\mathbb{R}^3}^2$ on the right hand side of (2.17), we write $\omega_a^+ = \omega_{(a,0)} + \omega^+$. The same formula also holds for the third components: $\omega_{a3}^+ = \omega_{(a,0),3} + \omega_3^+$. This yields $\omega_3^+ = \omega_{a3}^+ - \omega_{(a,0),3}$ and

$$\|(-\Delta)^{3/4}\omega_3^+\|_{2;\mathbb{R}^3}^2 \leq \|(-\Delta)^{3/4}\omega_{a3}^+\|_{2;\mathbb{R}^3}^2 + \|(-\Delta)^{3/4}\omega_{(a,0),3}\|_{2;\mathbb{R}^3}^2, \quad (3.9)$$

where

$$\begin{aligned} \|(-\Delta)^{3/4}\omega_{(a,0),3}\|_{2;\mathbb{R}^3}^2 &\leq \int_0^\infty d(F_\lambda A^{3/2}\omega_{(a,0)}, A^{3/2}\omega_{(a,0)})_{2;\mathbb{R}^3} \\ &= \int_0^\infty \lambda^3 d(F_\lambda \omega_{(a,0)}, \omega_{(a,0)})_{2;\mathbb{R}^3} = \int_0^\infty \lambda^3 d((E_\lambda - E_{-\lambda})\omega_{(a,0)}, \omega_{(a,0)})_{2;\mathbb{R}^3} \\ &= - \int_0^\infty \lambda^3 d(E_{-\lambda}\omega_{(a,0)}, \omega_{(a,0)})_{2;\mathbb{R}^3}. \end{aligned}$$

The last equality holds because $E_\lambda \omega_{(a,0)} = \omega_{(a,0)}$ for $\lambda > 0$, which means that $d(E_\lambda \omega_{(a,0)}, \omega_{(a,0)})_{2;\mathbb{R}^3} = 0$. Further, we have

$$- \int_0^\infty \lambda^3 d(E_{-\lambda}\omega_{(a,0)}, \omega_{(a,0)})_{2;\mathbb{R}^3} = - \int_0^{-a} \lambda^3 d(E_{-\lambda}\omega_{(a,0)}, \omega_{(a,0)})_{2;\mathbb{R}^3}$$

because $E_{-\lambda}\omega_{(a,0)} = \mathbf{0}$ for $-\lambda < a$, i.e. $\lambda > -a$. Using the substitution $\lambda = -\zeta$, the last integral transforms to

$$- \int_0^{-a} (-\zeta)^3 d(E_\zeta \omega_{(a,0)}, \omega_{(a,0)})_{2;\mathbb{R}^3} = \int_0^{|a_-|} \zeta^5 d(E_\zeta \mathbf{v}_{(a,0)}, \mathbf{v}_{(a,0)})_{2;\mathbb{R}^3} \leq c_5 |a_-|^5. \quad (3.10)$$

Using now (2.17), (3.9) and (3.10), we obtain the inequality

$$\|A^{1/2}\omega^+\|_{2;\mathbb{R}^3}^2 \leq c_3 \|\omega^+\|_{2;\mathbb{R}^3}^2 + c_4 \|(-\Delta)^{3/4}\omega_{a3}^+\|_{2;\mathbb{R}^3}^2 + c_4 c_5 |a_-|^5. \quad (3.11)$$

Both the right hand sides of (3.7) and (3.11) are integrable, as functions of variable t , on the interval $(0, T)$ due to condition (iv) of Theorem 2. The proof can now be again finished in the same way as the proof of Theorem 1 in Section 2.

4 Concluding remarks

Remark 4.1 (the meaning of functions \mathbf{v}^+ and ω^+). Using the spectral resolution of identity $\{E_\lambda\}$ associated with operator \mathbf{curl} , we can express velocity \mathbf{v} and the corresponding vorticity ω by the formulas

$$\mathbf{v} = \int_{-\infty}^{\infty} dE_\lambda(\mathbf{v}), \quad \omega = \int_{-\infty}^{\infty} \lambda dE_\lambda(\mathbf{v}) = \int_{-\infty}^{\infty} dE_\lambda(\omega). \quad (4.1)$$

In accordance with the heuristic understanding of the definite integral, we can interpret the first integral in (4.1) as a sum of “infinitely many” contributions $dE_\lambda(\mathbf{v})$, each of whose is an “infinitely small” Beltrami flow. (Recall that *Beltrami flows* are flows, whose vorticity is parallel to the velocity. Here, concretely, $\mathbf{curl} dE_\lambda(\mathbf{v}) = \lambda dE_\lambda(\mathbf{v})$.) Function \mathbf{v}^+ can now be understood to be the sum of only those “infinitely many” “infinitely small” contributions, whose vorticity is a positive multiple of velocity. (We call them the *positive Beltrami flows*.)

Remark 4.2 (flow in the neighbourhood of a singularity). Theorem 1 is also true if ω^+ (respectively ω_3^+) is replaced by ω^- (respectively ω_3^-). Thus, both the conditions (i) and (ii) show that if weak solution \mathbf{v} has a singular point then the singularity must contemporarily develop in the “positive part” \mathbf{v}^+ of function \mathbf{v} (the contribution to \mathbf{v} coming from the positive Beltrami flows) as well as in the “negative part” \mathbf{v}^- (the contribution from the negative Beltrami flows). The singularity must even develop at the same spatial point. (This can be proven by an appropriate localization procedure.)

Remark 4.3 (the role of large frequencies). Suppose, for simplicity, that function a considered in Theorem 2 is positive. Then projection P_a^+ defined by (1.9) can be interpreted as a reduction to the positive Beltrami flows with “high frequencies”, concretely the frequencies comparable to a and higher. Theorem 2 shows that if a singularity develops in solution \mathbf{v} , then it must especially develop in the part of \mathbf{v} (respectively its vorticity ω) that consists of positive Beltrami flows with the “large” frequencies (i.e. $\sim a$ and higher). Since the functions a_+ , ω_a^+ and ω_{a3}^+ can be replaced by a_- , ω_a^- and ω_{a3}^- in Theorem 2, the singularity must also develop in the part of \mathbf{v} (respectively vorticity ω) that consists of negative Beltrami flows with “large” frequencies. The singularities must appear in both the parts at the same space–time point.

Remark 4.4. If function a in Theorem 2 identically equals $-\infty$ in $(0, T)$ then $P_a^+ = I$ and $\omega_a^+ = \omega$ in $(0, T)$. In this case, condition (iii) is the condition on the whole vorticity ω , and it requires that $\omega \in L^2(0, T; D(S^{1/4}))$. (Recall that S is the Stokes operator in $\mathbf{L}_\sigma^2(\mathbb{R}^3)$.) The space $D(S^{1/4})$ is continuously imbedded in $\mathbf{L}^3(\mathbb{R}^3)$. Besides that, it is known that if $\omega \in L^2(0, T; \mathbf{L}^3(\mathbb{R}^3))$ then solution \mathbf{v} has no singular points in Q_T , see e.g. [1]. This comparison (made for $a \equiv -\infty$) gives hope that condition (iii) might be perhaps generalized so that it would only require $\omega_a^+ \in L^2(0, T; \mathbf{L}^3(\mathbb{R}^3))$ instead of $\omega_a^+ \in L^2(0, T; D(S^{1/4}))$ also for other functions a . Similar generalizations might also concern conditions (i), (ii) and (iv).

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Author's addresses:

Jiří Neustupa
 Czech Academy of Sciences
 Mathematical Institute
 Žitná 25, 115 67 Praha 1
 Czech Republic
 neustupa@math.cas.cz

Patrick Penel
 Université du Sud, Toulon–Var
 Dep. Mathématique
 BP 20132, 83957 La Garde
 France
 penel@univ-tln.fr